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Recently [G. Kaniadakis, Phys. Rev. A **55**, 941 (1997)], we introduced a Schrödinger equation containing a complex nonlinearity $W(\rho,\mathbf{j})+i\,\mathcal{W}(\rho,\mathbf{j})$ which describes the collective interaction introduced by an exclusion-inclusion principle (EIP). The EIP does not affect $W(\rho,\mathbf{j})$ and determines $\mathcal{W}(\rho,\mathbf{j})$ univocally. In the above reference $W(\rho,\mathbf{j})$ was deduced by means of a stochastic quantization approach, in this way obtaining a noncanonical quantum system. In this work we introduce a family of nonlinearities $W(\rho,\mathbf{j})$ generating a family of nonlinear canonical quantum systems, and derive their Lagrangian and the Hamiltonian functions and the evolution equations of the fields. We derive also the Ehrenfest relations and study the soliton properties. The shape of the soliton, propagating in the system obeying the EIP, can be obtained by solving a first-order ordinary differential equation. We show that, in the case of soliton solutions, by means of a unitary transformation, the EIP potential is equivalent to a real algebraic nonlinear potential proportional to $\kappa\,\rho^2/(1+\kappa\,\rho)$. [S1063-651X(98)11511-8]

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I. INTRODUCTION

It is widely known that the effects of collective interactions in a particle system, commonly studied in the framework of quantum many-body theories, can be linked to nonlinearities in the one-particle Schrödinger equation. Several nonlinear Schrödinger equations (NLSE's) have been studied in the past, and recently, they have been commonly used in many different fields of research in physics. The cubic equation, for instance, with the nonlinear term proportional to $\pm|\psi|^2$, has been used to study the dynamical evolution of a boson gas with a δ-function pairwise repulsion or attraction, responsible for its anyonlike behavior [1]. Recently, this equation has been used to describe the Bose-Einstein condensation [2–6] and the dynamics of two-dimensional radiating vortices [7]. The nonlinear term $|\psi|^2$ appears in the Ginzburg-Landau model of the superconductivity [8], a phenomenon also investigated by means of the Eckhaus equation, which is a NLSE with a nonlinearity of the type $|\psi|^2+\alpha|\psi|^4$ [9]. The same equation appears in superfluidity, where the properties of a gas of bosons interacting via a two-body attractive and three-body repulsive δ function interparticle potential are investigated [10,11]. The Eckhaus equation can describe nonlinear waves in optical fibers with a ''normal'' dependence of the refractive index on the light intensity [7]. NLSE's with nonlinearities involving the quantity $\mathbf{j}$ have been also introduced to study planar systems of particles with anyonlike unconventional statistics [12]. In the literature we can find NLSE's with complex- or derivative-type nonlinearities involving the quantities $(\boldsymbol{\nabla}\rho)^2, \Delta\rho, \mathbf{j}\cdot\boldsymbol{\nabla}\rho, \boldsymbol{\nabla}\mathbf{j}$ [13–16] as, for instance, in the Doebner-Goldin equation associated with a certain unitary group representation and describing irreversible and dissipative quantum systems.

In a one-dimensional particle system, a firm relation exists between statistics and collective interactions. For instance, the potential $U(x_i)=m^{-1}\alpha(\alpha-1)\ \sum_{j<i}(x_i-x_j)^{-2}$, describing the collective interaction of the Calogero-Sutherland model, implies that the particle system is ruled by the Haldane-Wu statistics [17–19].

Also a two-dimensional system, where collective excitations can describe superconductive features, the particles composing the system obey the so called anyonic statistics, actually an important advanced field of research. Collective excitations, usually studied by means of a many body quantum theory, can also be analyzed with the one-particle NLSE approach. Due to this fact, we can argue that an eventually nonunivocal relation holds between statistics and interaction introduced by the nonlinearities contained in NLSE's. For instance, in the case of the anyons, many authors adopt the NLSE approach to study the superconductivity.

Since the beginning of quantum mechanics (1932) [20], it was understood that effects due to the statistics and imposed by the Pauli exclusion principle to a system of free fermions can be simulated by a repulsive potential in the coordinate space; under its action the particles evolve in time. Analogously, free bosons can be submitted to an attractive potential.

When we deal with many-body fermion systems, due to the presence and effectiveness of the Pauli exclusion principle, we may encounter serious difficulties in calculating the dynamics and the stationary states. After the introduction of particles obeying intermediate statistics and of a generalized Pauli principle, the difficulties can be increased. A semiclassical approach to describe systems of particles of different intermediate statistics, from Fermi-Dirac to Bose-Einstein, is

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very useful in deriving evolution and equilibrium statistical distributions. The exclusion-inclusion principle (EIP) (as we call a generalized Pauli principle acting in the coordinate space) can be accounted for, within a semiclassical approach, to describe systems of particles with different statistics by using Fokker-Planck and/or Boltzmann equations.

Recently [21], by using an appropriate stochastic quantization method, we have quantized a Markoffian process which obeys a generalized Pauli exclusion-inclusion principle in the coordinate space, and have obtained the following NLSE:

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\Delta\psi+\Lambda(\rho,\mathbf{j})\psi+V\psi, \qquad (1)$$

where

$$\Lambda(\rho,\mathbf{j})=W(\rho,\mathbf{j})+i\,\mathcal{W}(\rho,\mathbf{j}), \qquad (2)$$

is the complex nonlinearity introduced by the EIP, and the expression of the imaginary part is

$$\mathcal{W}(\rho,\mathbf{j})=-\kappa\frac{\hbar}{2\rho}\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{j}\rho}{1+\kappa\rho}\right). \qquad (3)$$

The expression of the quantum current is

$$\mathbf{j}=-\frac{i\hbar}{2m}(1+\kappa\rho)(\psi^*\boldsymbol{\nabla}\psi-\psi\boldsymbol{\nabla}\psi^*). \qquad (4)$$

The EIP enters the expression of $\mathbf{j}$ through the multiplicative factor $1+\kappa\rho$, which behaves as an enhancement factor when $\kappa>0$, and as an inhibition factor when $\kappa<0$, while the EIP is absent when $\kappa=0$. Since the above factor depends on $\rho=|\psi|^2$, the EIP introduces a collective interaction. In Ref. [21] it was shown that the parameter κ is the lower bound, and the allowed range of its values is $\kappa\geqslant-1/\rho_{max}$. The current $\mathbf{j}$ in Eq. (4) satisfies the continuity equation

$$\frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}\cdot\mathbf{j}=0. \qquad (5)$$

The real part of the complex term is not determined by the EIP. There are infinite choices of the real part of the nonlinearity $\Lambda(\rho,\mathbf{j})$; any choice does not violate the EIP.

We recall that in Ref. [21], in order to obtain intermediate statistics, we used a stochastic quantization approach [22] and derived the following expression of the real part of the nonlinear term:

$$W(\rho)=\kappa\frac{\hbar^2}{4m}\left[\frac{\Delta\rho}{1+\kappa\rho}+\frac{2-\kappa\rho}{2\rho(1+\kappa\rho)^2}(\boldsymbol{\nabla}\rho)^2\right]. \qquad (6)$$

It can be shown that the system described by Eqs. (1), (2), (3), and (6) is noncanonical. Its noncanonicity is due to the particular choice of the nonlinearity given by Eq. (6).

The classical motion of canonical coordinates and momenta has a simple quantum analog. For this reason quantum mechanics has been built up in analogy with the classical Hamiltonian theory. The Lagrangian allows one to collect all the equations of motion and express them as stationary properties of a given action function.

Because the approach of stochastic quantization is noncanonical, we think that it is of great importance for its applications to verify if it can be also canonically formulated and to derive a canonical formulation.

The introduction of the EIP is equivalent to the insertion into the Hamiltonian function of a potential which sets a collective behavior to the particles of the system and can be added to other potentials that take into account the interactions of particles among themselves. Let us discuss two examples where EIP can be usefully applied. In nuclear physics the correlation effects between pairs of nucleons, viewed as fermions, are quite relevant in the interpretation of experimental results. Similarly, the interactions among bosons are relevant in various nuclear modes (superfluid model, interacting boson model, mean field boson approximation) and allow the explanation of many collective nuclear properties. The interaction among the fermionic valence nucleons outside the core produces pairs of correlated nucleons that can be approximated as particles with a behavior intermediate between fermionic and bosonic ones. This nuclear state (quasideuteron state) can be viewed as a particle system which obeys to EIP. Recently, we studied a semiclassical model of photofission in the quasideuteron energy region [23]. We described the quasideuteron state as a mixture of fermion and boson states, with a good agreement of our calculated photofission cross sections of several heavy nuclei and experimental results.

The second example is the Bose-Einstein condensation (BEC): The condensation originates from an attraction of statistical nature (Bose-Einstein statistics) among the particles. In several papers BEC is studied by means of a cubic NLSE [2]. In place of the cubic (and simplest) nonlinearity, other nonlinearities can be considered as, for instance, the one introduced by EIP with a positive κ parameter to simulate the particle attraction.

We consider the present work as a natural continuation of Ref. [21]. We examine canonical systems that obey the EIP, and analyze aspects of the Lagrangian and Hamiltonian formalism, the Ehrenfest relations, and the solitonic properties of the canonical systems.

In Sec. II, we recall the main relations of the canonical formulation that will be used in this work. In Sec. III, after the introduction of the Lagrangian density, we derive a nonlinear equation which contains, within the canonical formulation, all the effects introduced by the exclusion-inclusion principle. The Hamiltonian formulation is also developed. In Sec. IV, we fix our attention to the hydrodynamic formulation of the canonical system obeying the EIP, and derive the evolution equations of the fields ρ and S. In Sec. V we discuss the Eherenfest relations and write the mean conserved quantities of the EIP system; successively, in Sec. VI, we study the solitonlike solutions of our system. In Sec. VII we derive an effective potential simulating the effects introduced by the EIP in the case of solitons, and we rewrite the equations describing the shape of the solitons in a form which allows us to use the soliton techniques available in the literature. In Sec. VIII, as an application, we consider a system where the collective effects, are described by a particular nonlinearity, and derive explicitly the shape and the phase of the soliton profile. Conclusions are reported in Sec. IX.

II. CANONICAL FORMALISM OUTLOOK

Let us recall the main quantities, definitions, and relations of the canonical formulation that we will use in this work. We consider a quantum N-field nonrelativistic canonical system described by the Lagrangian density $\mathcal{L}[\phi_j] \equiv \mathcal{L}(\phi_j, \partial_t \phi_j, \partial_i \phi_j)$ function of the scalar fields $\phi_j(\mathbf{x},t)$ and of their first derivatives $\partial_t \phi_j$ and $\partial_i \phi_j$ with $\phi_j \in L^2$, $j = 1,\ldots,N$; $\partial_t = \partial/\partial t$; and $\partial_i = \partial/\partial x_i$, where $i=1,2$, and 3. The Lagrangian function L of the system is the functional

$$L = \int \mathcal{L}\, d^3x, \tag{7}$$

and the evolution equations of the fields ϕ_j can be obtained by the variational principle

$$\frac{\delta L}{\delta \phi_j} = 0, \quad j = 1,\ldots,N, \tag{8}$$

where the functional derivative is defined as

$$\frac{\delta L}{\delta \phi_j} = \frac{\partial \mathcal{L}}{\partial \phi_j} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial(\partial_t \phi_j)} - \sum_{i=1}^{3} \frac{d}{dx_i}\frac{\partial \mathcal{L}}{\partial(\partial_i \phi_j)}. \tag{9}$$

If we introduce the fields π_j, canonically conjugate momenta of the fields ϕ_j defined as

$$\pi_j = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi_j)}, \tag{10}$$

we can write the Hamiltonian density $\mathcal{H}[\phi_j] \equiv \mathcal{H}(\phi_j, \pi_j, \partial_i \phi_j, \partial_i \pi_j)$, related to the Lagrangian density, by a Legendre transformation

$$\mathcal{H} = \sum_{j=1}^{N} \pi_j \frac{\partial \phi_j}{\partial t} - \mathcal{L}. \tag{11}$$

The Hamiltonian H of the system is

$$H = \int \mathcal{H}\, d^3x. \tag{12}$$

The system admits a canonical formulation if the evolution with time of the fields ϕ_j and of the canonically conjugate momenta π_j can be described by the Hamilton equations (see, for example, Ref. [24])

$$\frac{\partial \phi_j}{\partial t} = \frac{\delta H}{\delta \pi_j}, \tag{13}$$

$$\frac{\partial \pi_j}{\partial t} = -\frac{\delta H}{\delta \phi_j}, \tag{14}$$

that can be written in a fully equivalent way, by means of the Poisson formalism:

$$\frac{\partial \phi_j}{\partial t} = \{\phi_j, H\}, \tag{15}$$

$$\frac{\partial \pi_j}{\partial t} = \{\pi_j, H\}, \tag{16}$$

where the Poisson brackets of the fields (or functions) $A(\mathbf{x}) = A[\phi_j(\mathbf{x}), \pi_j(\mathbf{x})]$ and $B(\mathbf{y}) = B[\phi_j(\mathbf{y}), \pi_j(\mathbf{y})]$ are defined by the relation

$$\{A(\mathbf{x}), B(\mathbf{y})\} = \sum_{j=1}^{N} \int \left[\frac{\delta A(\mathbf{x})}{\delta \phi_j(\mathbf{z})} \frac{\delta B(\mathbf{y})}{\delta \pi_j(\mathbf{z})} - \frac{\delta B(\mathbf{y})}{\delta \phi_j(\mathbf{z})} \frac{\delta A(\mathbf{x})}{\delta \pi_j(\mathbf{z})} \right] d^3z. \tag{17}$$

III. CANONICAL EIP SYSTEM

In this section, we will assume a particular expression of the Lagrangian density, and show *a posteriori* that the equation of motion admits the continuity equation (5) with the current $\mathbf{j}$ given by Eq. (4). Let us consider the complex function ψ, corresponding to the field used to describe the quantum system, and derive the differential equations that the function ψ and its complex conjugate ψ^* must satisfy. The Lagrangian density $\mathcal{L}$ can be expressed in terms of the fields ψ and ψ^*:

$$\mathcal{L} = i\hbar\, \psi^* \frac{\partial \psi}{\partial t} - \frac{\hbar^2}{2m} \nabla \psi^* \nabla \psi - V \psi^* \psi - U(\psi^* \psi) + \kappa \frac{\hbar^2}{8m} (\psi^* \nabla \psi - \psi \nabla \psi^*)^2. \tag{18}$$

The first three terms in the Lagrangian density are the same encountered in the standard linear quantum description. The fourth term is a nonlinear real potential that, for the sake of simplicity, we assume to be an arbitrary analytic nonderivative function only of $\rho = \psi^* \psi$. This potential is used to describe other interactions in the system. By appropriately choosing its form, Lagrangian (18) can be used to describe different physical systems containing nonlinear interactions and admitting the EIP (see below, Sec. VIII). Finally, the last term is the EIP potential

$$U_{\rm EIP}(\psi, \psi^*) = -\kappa \frac{\hbar^2}{8m} (\psi^* \nabla \psi - \psi \nabla \psi^*)^2; \tag{19}$$

using Eq. (4), we can write

$$U_{\rm EIP}(\rho, \mathbf{j}) = \kappa \frac{m}{2} \left(\frac{\mathbf{j}}{1 + \kappa \rho} \right)^2. \tag{20}$$

The Lagrangian $L = L[\psi^*, \psi]$ is given by the Eqs. (7) and (18). The Schrödinger equation for the field ψ can be obtained by requiring that the functional derivative of L with respect to ψ^* should be zero:

$$\frac{\delta L}{\delta \psi^*} = 0, \tag{21}$$

and can be written, in an explicit form, as

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\Delta\psi+V\psi+F(\rho)\psi-\kappa\frac{\hbar^2}{2m}(\psi^*\nabla\psi-\psi\nabla\psi^*)\nabla\psi-\kappa\frac{\hbar^2}{4m}\nabla(\psi^*\nabla\psi-\psi\nabla\psi^*)\psi, \tag{22}$$

where $F(\rho)$ is given by

$$F(\rho)=\frac{\partial U(\rho)}{\partial\rho}. \tag{23}$$

It is easy to see that this nonlinear evolution equation is consistent with the continuity equation (5) with the current given by Eq. (4). This equation can also be written, using the current (4) and the particle density ρ, in the form

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\Delta\psi+V\psi+F(\rho)\psi+\kappa\frac{m}{\rho}\left(\frac{\mathbf{j}}{1+\kappa\rho}\right)^2\psi-i\kappa\frac{\hbar}{2\rho}\nabla\left(\frac{\mathbf{j}\rho}{1+\kappa\rho}\right)\psi. \tag{24}$$

Equation (24) is the NLSE of Eq. (1), with the same imaginary part $\mathcal{W}(\rho,\mathbf{j})$ and with the real part $W(\rho,\mathbf{j})$ of the nonlinear quantity $\Lambda(\rho,\mathbf{j})$ given by

$$W(\rho,\mathbf{j})=\kappa\frac{m}{\rho}\left(\frac{\mathbf{j}}{1+\kappa\rho}\right)^2. \tag{25}$$

Analogously, the Schrödinger equation for the field ψ^* can be explicitly derived from the relation $\delta L/\delta\psi=0$.

Note that the imaginary part of Eq. (24) is fixed by the requirement that the system described by Lagrangian (18) satisfies the continuity equation (5) with the current (4), which we write as

$$\mathbf{j}=(1+\kappa\rho)\mathbf{j}_0, \tag{26}$$

where

$$\mathbf{j}_0=-\frac{i\hbar}{2m}(\psi^*\nabla\psi-\psi\nabla\psi^*) \tag{27}$$

is the quantum current without the EIP. The real part of the nonlinearity $\Lambda(\rho,\mathbf{j})$, given by Eq. (25), is obtained from the requirement that the system must be canonical. This expression is different from that obtained in Ref. [21], where the system is considered noncanonical and the form of $W(\rho,\mathbf{j})$ is derived by using the stochastic quantization method.

The quantum system described by the Lagrangian density (18) is canonical. This can be verified defining the field π_ψ, canonically conjugate to the field ψ, by means of relation (10):

$$\pi_\psi=i\hbar\psi^*. \tag{28}$$

It is well known that π_ψ is proportional to the field ψ^* and, while in the Lagrangian formalism ψ and ψ^* are independent fields, in the Hamiltonian formalism they are canonically conjugate. Let us also note that $\pi_{\psi^*}=0$ and the degrees of freedom are the same in both the Lagrangian and Hamiltonian formalisms.

Performing the Legendre transformation (11), it is easy to see that the Hamiltonian density can be written as

$$\mathcal{H}=\frac{\hbar^2}{2m}\nabla\psi^*\nabla\psi+V\psi^*\psi+U(\psi^*\psi)-\kappa\frac{\hbar^2}{8m}(\psi^*\nabla\psi-\psi\nabla\psi^*)^2. \tag{29}$$

From Eq. (29), it appears that the system may be unstable if $\kappa<-1/\rho_{\max}$. This is because if we ignore the potentials involving V and U, modes with wave number $\mathbf{k}$ [$\psi=\sqrt{\rho_{\max}}\exp\{i(\mathbf{k}\cdot\mathbf{x}-\omega t)\}$] have an energy $E=(\hbar^2\mathbf{k}^2/2m)(1+\kappa\rho_{\max})$ which must be positive. This imposes the condition $\kappa\geqslant-1/\rho_{\max}$, that is the same as that obtained in Ref. [21].

The Hamilton equations take the forms

$$i\hbar\frac{\partial\psi}{\partial t}=\frac{\delta H}{\delta\psi^*}, \tag{30}$$

$$-i\hbar\frac{\partial\psi^*}{\partial t}=\frac{\delta H}{\delta\psi}, \tag{31}$$

that are the Schrödinger equations for the fields ψ and ψ^*, respectively. Within the Poisson formalism, Eqs. (30) and (31) assume the form

$$\frac{\partial\psi}{\partial t}=\{\psi,H\}, \tag{32}$$

$$\frac{\partial\psi^*}{\partial t}=\{\psi^*,H\}, \tag{33}$$

as can be easily verified.

The physical system of particles obeying to the EIP introduced semiclassically can be described by a canonical formulation. This fact is of fundamental importance to the theoretical ground and to the possible applications.

IV. HYDRODYNAMIC FORMULATION

The wave function ψ can be expressed in terms of the particle density ρ and the phase S:

$$\psi(\mathbf{x},t)=\rho(\mathbf{x},t)^{1/2}\exp\left[\frac{i}{\hbar}S(\mathbf{x},t)\right]. \tag{34}$$

We describe the physical EIP system in terms of these two real independent fields [25] (see also Ref. [26], where a nonlinear Schrödinger equation is derived from a variational principle in the ρ-S representation).

Taking into account Eqs. (24) and (34), we obtain the evolution equations of the phase S and of the particle density ρ:

$$\frac{\partial S}{\partial t}+\frac{(\nabla S)^2}{2\,m}-\frac{\hbar^2}{2\,m}\,\frac{\Delta\rho^{1/2}}{\rho^{1/2}}+V+\kappa\,\rho\,\frac{(\nabla S)^2}{m}+F(\rho)=0, \tag{35}$$

$$\frac{\partial \rho}{\partial t}+\nabla\left[\frac{\nabla S}{m}\,\rho\,(1+\kappa\,\rho)\right]=0. \tag{36}$$

Equation (35) is a *Hamilton-Jacobi*-type equation, where the third term is the quantum potential [25]. The fifth term is exactly the real part $W(\rho,\mathbf{j})$ of the term introduced by the EIP potential (19), as can be easily verified keeping in mind the expression of the current $\mathbf{j}$ given by Eq. (4), or by

$$\mathbf{j}=\frac{\nabla S}{m}\,\rho\,(1+\kappa\,\rho). \tag{37}$$

Finally, the last term in Eq. (35) is the extra nonlinearity discussed in Sec. III. We can see that the imaginary part $\mathcal{W}(\rho,\mathbf{j})$ of the term induced by the EIP potential does not appear in the Hamilton-Jacobi equation (35). The expression of the current (37) is immediately obtained if we take into account Eq. (36), that is the continuity equation in the ρ-S representation (the integral $\int\rho\,d^3x$ is conserved).

The Hamilton-Jacobi equation (35) and the continuity equation (36) can be derived from a variational principle, removing the two functional derivatives of the Lagrangian $\tilde{L}[\rho,S]$:

$$\frac{\delta\tilde{L}}{\delta\rho}=0,\quad \frac{\delta\tilde{L}}{\delta S}=0. \tag{38}$$

The Lagrangian $\tilde{L}$ is defined in Eq. (7) starting from the Lagrangian density $\tilde{\mathcal{L}}$, which is given by

$$\tilde{\mathcal{L}}=i\,\frac{\hbar}{2}\,\frac{\partial\rho}{\partial t}-\frac{\partial S}{\partial t}\,\rho-\frac{(\nabla S)^2}{2\,m}\,\rho-\frac{\hbar^2}{8\,m}\,\frac{(\nabla\rho)^2}{\rho}-V\,\rho-U(\rho)-\kappa\,\frac{(\nabla S)^2}{2\,m}\,\rho^2, \tag{39}$$

as can be easily verified taking into account Eqs. (18) and (34). In the Lagrangian density (39) the first five terms are the same terms occurring in the linear Schrödinger equation, while the nonlinear contribution is given by the other terms, the last of this is the potential introduced by the EIP.

Let us now introduce the Hamiltonian procedure. The momentum π_S, canonically conjugate to S, is given by Eq. (10), that now becomes

$$\pi_S=-\rho. \tag{40}$$

Moreover, we have $\pi_\rho=i\,\hbar/2$. Therefore, as in Sec. III, π_S is proportional to ρ, while π_ρ is a constant; the number of degrees of freedom is the same in both the Lagrangian and Hamiltonian formalisms.

The Hamiltonian density, a function of the canonically conjugate fields S and $-\rho$, can be deduced taking into account Eqs. (11) and (39):

$$\tilde{\mathcal{H}}=\frac{(\nabla S)^2}{2\,m}\,\rho+\frac{\hbar^2}{8\,m}\,\frac{(\nabla\rho)^2}{\rho}+V\,\rho+U(\rho)+\kappa\,\frac{(\nabla S)^2}{2\,m}\,\rho^2. \tag{41}$$

The Hamilton-Jacobi and continuity equations take the forms

$$\frac{\partial S}{\partial t}=-\frac{\delta\tilde{H}}{\delta\rho}, \tag{42}$$

$$\frac{\partial\rho}{\partial t}=\frac{\delta\tilde{H}}{\delta S}. \tag{43}$$

The same equations, in the Poisson formalism, can be rewritten as

$$\frac{\partial S}{\partial t}=\{S,\tilde{H}\}, \tag{44}$$

$$\frac{\partial\rho}{\partial t}=\{\rho,\tilde{H}\}. \tag{45}$$

The evolution equations (30) and (31) or (32) and (33), deduced from $H[\psi,\psi^*]$, have the same forms as the Eq. (42) and (43) or (44) and (45) deduced from $\tilde{H}[\rho,S]$. Then, according to a well-established procedure, we can relate the fields ψ-ψ^* to the fields S-ρ by means of a canonical transformation [27]. The equations of motion in the S-ρ representation will be used in Sec. VI to study particular soliton solutions of Eq. (24) that preserve their shapes in the time. As we will show, we are able to decouple the system of equations (42) and (43) or equivalently Eqs. (44) and (45), obtaining a differential equation in the variable ρ only, whose solutions define the solitons of the systems with the EIP.

V. EHRENFEST RELATIONS

We discuss the effect of the EIP on the time evolution of the average of the most important physical observable that describe the system. Let us assume that the nonlinear potential $U(\rho)$ and the field ψ vanish at infinity, so that the surface terms can be disregarded. Moreover, we assume that the potential $U(\rho)$ depends on the space and on the time only, through the field ρ.

To obtain the Ehrenfest relations of the system obeying to Eq. (24) we verify, first that for a Hermitian operator $\hat{A}=\hat{A}^\dagger$ the following relation holds:

$$\frac{d}{d\,t}\langle\hat{A}\rangle=\frac{i}{\hbar}\int\left[\frac{\delta H}{\delta\psi}\hat{A}\,\psi-\psi^*\,\hat{A}\frac{\delta H}{\delta\psi^*}\right]d^3x+\left\langle\frac{\partial\hat{A}}{\partial t}\right\rangle. \tag{46}$$

Let us call $\hat{O}$ the operator on the right hand side of the NLSE (24), that can be rewritten in the form

$$i\,\hbar\,\frac{\partial\psi}{\partial t}=\hat{O}\,\psi, \tag{47}$$

with

$$\hat{O}=\hat{H}_0+F(\rho)+W(\rho,\mathbf{j})+i\,\mathcal{W}(\rho,\mathbf{j}), \tag{48}$$

where $\hat{H}_0=(-\hbar^2/2\,m)\,\Delta+V(\mathbf{x})$ is the Hamiltonian operator of the linear theory. We can write relation (46) in the form

$$\frac{d}{d\,t}\langle\hat{A}\rangle=\frac{i}{\hbar}\langle[\mathrm{Re}\,\hat{O},\hat{A}]\rangle+\frac{1}{\hbar}\langle\{\mathrm{Im}\,\hat{O},\hat{A}\}\rangle+\left\langle\frac{\partial\hat{A}}{\partial\,t}\right\rangle, \tag{49}$$

where the symbols $[\,\cdot\,,\cdot\,]$ and $\{\,\cdot\,,\cdot\,\}$ indicate the commutator and anticommutator, respectively (also see Ref. [14], where an example of the Ehrenfest relations in a nonlinear Schrödinger equation with complex potential is discussed).

After posing $\hat{A}=\mathbf{x}$ in Eq. (49), we obtain the following Ehrenfest relation:

$$\frac{d}{d\,t}\langle\mathbf{x}\rangle=\frac{1}{m}\langle(1+\kappa\,\rho)\,\hat{\mathbf{p}}\rangle. \tag{50}$$

It is worth remarking that the EIP introduces the additional momentum $\langle\kappa\,\rho\,\hat{\mathbf{p}}\rangle=\kappa\,m\,\mathbf{j}/(1+\kappa\,\rho)$. Note also that the right hand side of Eq. (50) can be written as $\int\mathbf{j}\,d^3x$, as in the standard linear quantum mechanics.

A second relation is obtained setting $\hat{A}=\hat{\mathbf{p}}$:

$$\frac{d}{d\,t}\langle\hat{\mathbf{p}}\rangle=-\langle\boldsymbol{\nabla}\,V\rangle-\langle\boldsymbol{\nabla}F\rangle. \tag{51}$$

Equation (51) can be seen as the second law of the dynamics [28,29]. The dynamics of the mean value of the momentum is governed by an *effective potential* given by the sum of the external potential V and of the nonlinearity $F(\rho)$. The EIP potential does not affect, on the average, the dynamics of the system because, due to their particular form, the terms W and $\mathcal{W}$ satisfy the relation $\langle[W,\boldsymbol{\nabla}]-i\,\{\mathcal{W},\boldsymbol{\nabla}\}\rangle=0$. For the most frequent nonlinearity $F(\rho)$, generally appearing in the nonlinear Schrödinger equations, the last term can be dropped and the Newtonian behavior is restored [28]. Conversely, other dynamical equations, like the sine-Gordon equation, do seem to show a different behavior from the Newtonian one.

Next, setting $\hat{A}=\hat{\mathbf{L}}$, where $\hat{\mathbf{L}}$ is the angular momentum operator with components $\hat{L}_i=\varepsilon_{ijk}\,x_j\,\hat{p}_k$, we obtain

$$\frac{d}{d\,t}\langle\hat{\mathbf{L}}\rangle=\langle\mathbf{M}\rangle+\langle\mathbf{M}'\rangle, \tag{52}$$

with $\mathbf{M}$, the momentum of the external force ($M_i=-\varepsilon_{ijk}\,x_j\,\partial_k V$), and $\mathbf{M}'$, the momentum of the internal force ($M'_i=-\varepsilon_{ijk}\,x_j\,\partial_k F$), introduced by the potential $U(\rho)$. As in the previous relation, the EIP potential does not contribute to the average of the angular momentum. Again if the nonlinear potential has a well behavior at infinity, the last term is unremarkable on the dynamics of the system.

We discuss now the Ehrenfest relation concerning the energy. Let us note that the energy of a nonlinear, noncanonical system is defined, analogously to the linear theories, as $E=\langle\hat{O}\rangle$, where $\hat{O}$ is the operator of the right hand side of the NLSE (24). The energy of a canonical system, as we are considering here, is given by $E=H$ where H is the Hamiltonian given by Eqs. (12) and (29). We can define a Hamiltonian operator $\hat{H}$ whose average value is $\langle\hat{H}\rangle=H$. It is easily verified that

$$\hat{H}=-\frac{\hbar^2}{2\,m}\Delta+V(\mathbf{x},t)+\frac{1}{\rho}U(\rho)+\frac{1}{\rho}U_{\mathrm{EIP}}(\rho,\mathbf{j}). \tag{53}$$

If we compare this expression of $\hat{H}$ with the expression of the operator $\hat{O}$ given by Eq. (48), we find

$$\hat{H}\neq\hat{O}, \tag{54}$$

meaning that the Hamiltonian operator of a nonlinear canonical system does not coincide with the operator $\hat{O}$ of the right hand side of the NLSE. Of course, in the case of the linear theories we have $\hat{O}=\hat{H}=\hat{H}_0$.

Within the definition $E=\langle\hat{H}\rangle$, in Appendix A we show the following relation:

$$\frac{d\,E}{d\,t}=\left\langle\frac{\partial\,V}{\partial\,t}\right\rangle, \tag{55}$$

meaning that when time dependent external potentials are absent, the system is conservative, being $d\,E/d\,t=0$. We may conclude that the EIP does not introduce dissipative effects. Of course we have $\langle\mathrm{Im}\,\hat{O}\rangle=\langle\mathcal{W}\rangle=0$ and then $d\langle\hat{O}\rangle/d\,t=d\langle\mathrm{Re}\,\hat{O}\rangle/d\,t\neq0$; the quantity $\langle\hat{O}\rangle$ is not conserved.

Several models with nonlinearities on the right hand side of the Schrödinger equation, characterized by time independent average values, have been developed. For instance, in the Kostin NLSE [30,31], the operator $\hat{O}$ is defined as $\hat{O}=\hat{H}_0+(\hbar\,\gamma/2\,i)\,[\log(\psi/\psi^*)-\langle\log(\psi/\psi^*)\rangle]$ being a real quantity; the energy of the system is defined as $E=\langle\hat{O}\rangle$. In this case the nonconservation of $\langle\hat{O}\rangle$ implies energy dissipation of the system.

In conclusion, we have shown that the EIP potential (19) describes a conservative system. For a free system ($V=0$) and when the nonlinear potential $U(\rho)$ has a good behavior at infinity, we are able to identify four constants of motion:

$$E=\int\mathcal{H}\,d^3x, \tag{56}$$

$$\langle\hat{\mathbf{p}}\rangle=-i\,\hbar\int\psi^*\,\boldsymbol{\nabla}\,\psi\,d^3x, \tag{57}$$

$$\langle\hat{\mathbf{L}}\rangle=-i\,\hbar\int\psi^*\,\mathbf{x}\times\boldsymbol{\nabla}\,\psi\,d^3x, \tag{58}$$

$$N=\int\rho\,d^3x, \tag{59}$$

representing, respectively, the energy, momentum, angular momentum, and number of particles, conserved in virtue of the continuity equation (5).

VI. SOLITONS

In this section we study a particular class of solutions of Eq. (24). In the free case, i.e., when $V=0$, we can consider the motion of the mass center on a straight line with uniform velocity (for a discussion of solitons in an external potential see, for example, Ref. [32]). In addition to the condition $V=0$, we limit our attention to the one-dimensional case when the EIP holds. To obtain solitonic solutions we use the method described in Ref. [33], valid for the NLSE's that are most frequently encountered in physical problems. Assuming that the field ψ depends only on the time t and on the coordinate of the soliton mass center $\xi=x\mp u\,t$ (as usual, the minus sign stands for a soliton moving from the left to the right side of the x axis, while the plus sign stands for an antisoliton moving in the opposite way) the wave function becomes $\psi(\mathbf{x},t)\equiv\psi(\xi,t)$ and the particle density $\rho(\mathbf{x},t)\equiv\rho(\xi)$. We assume that, for $\xi\to\pm\infty$, the particle density $\rho(\xi)\to 0$ so that $\int_{-\infty}^{+\infty}\rho(\xi)\,d\xi=N$, where N represents the particle number of the system. Moreover, the phase S can be written as

$$S=s(\xi)-\epsilon\,t, \tag{60}$$

and the field ψ assumes the form

$$\psi=\rho(\xi)^{1/2}\exp\left\{\frac{i}{\hbar}[s(\xi)-\epsilon\,t]\right\}. \tag{61}$$

The parameter ϵ is related to the parameter ε of Ref. [34] through the relation $\epsilon=-\varepsilon-m\,u^2/2$. It is now easy to verify that the Hamilton-Jacobi equations (35) and the continuity equation (36) describing the solitonic state can be reduced to the following system of coupling equations

$$\pm u\,\frac{\partial s}{\partial \xi}=\frac{1+2\,\kappa\,\rho}{2\,m}\left(\frac{\partial s}{\partial \xi}\right)^2+V_q(\rho)+F(\rho)-\epsilon, \tag{62}$$

$$\pm u\,\frac{\partial \rho}{\partial \xi}=\frac{1}{m}\frac{\partial}{\partial \xi}\left[\frac{\partial s}{\partial \xi}\,\rho\,(1+\kappa\,\rho)\right], \tag{63}$$

where we have indicated with $V_q(\rho)$ the one-dimensional quantum potential in the ξ coordinate:

$$V_q(\rho)=-\frac{\hbar^2}{2\,m}\,\frac{1}{\rho^{1/2}}\,\frac{\partial^2\rho^{1/2}}{\partial\xi^2}. \tag{64}$$

The quantum velocity $v_q(\xi)=m^{-1}\partial s(\xi)/\partial\xi$ must be finite when $\xi\to\pm\infty$; with this condition, Eq. (63) can be integrated a first time, obtaining

$$\frac{\partial\, s}{\partial\, \xi}=\pm\frac{m\,u}{1+\kappa\,\rho}, \tag{65}$$

and after second integration with the condition $\xi(0)=0$, we have

$$s(\xi)=\pm m\,u\int_0^{\xi}\frac{d\,\xi'}{1+\kappa\,\rho(\xi')}. \tag{66}$$

Equations (60) and (66) allow us to calculate the phase $S(\xi)$, provided that the quantity $\rho(\xi)$ is known.

To evaluate the density $\rho(\xi)$ we note that, if we take into account Eq. (65), Eq. (62) reduces to the following second-order differential equation:

$$\frac{2}{\rho}\,\frac{d^2\rho}{d\,\xi^2}-\left(\frac{1}{\rho}\,\frac{d\,\rho}{d\,\xi}\right)^2+\frac{(2\,m\,u/\hbar)^2}{(1+\kappa\,\rho)^2}-\frac{8\,m}{\hbar^2}F(\rho)+\frac{8\,m\,\epsilon}{\hbar^2}=0. \tag{67}$$

Before solving this equation, we observe that ϵ can be written in the form

$$\epsilon=V_q+F(\rho)-\frac{1}{2}m\,u^2\frac{1}{(1+\kappa\,\rho)^2}. \tag{68}$$

Equation (68) has an immediate physical interpretation: the quantum potential causes the spreading of the ordinary Schrödinger wave packet; this spreading is compensated for by the nonlinearity $F(\rho)$ and by the EIP contribution $-(m\,u^2/2)\,(1+\kappa\,\rho)^{-2}$. Therefore it is possible to build up a nonspreading solitary wave.

After the introduction of the function

$$y(\rho)=\left(\frac{1}{\rho}\,\frac{d\,\rho}{d\,\xi}\right)^2, \tag{69}$$

Eq. (67) reduces to a first-order linear differential equation

$$\frac{d\,y}{d\,\rho}+\frac{y}{\rho}+\frac{(2\,m\,u/\hbar)^2}{\rho\,(1+\kappa\,\rho)^2}-\frac{8\,m}{\hbar^2}\,\frac{F(\rho)}{\rho}+\frac{8\,m\,\epsilon}{\hbar^2}\,\frac{1}{\rho}=0, \tag{70}$$

that can be easily integrated, giving

$$y(\rho)=\frac{A}{\rho}-\frac{8\,m\,\epsilon}{\hbar^2}+\frac{(2\,m\,u/\hbar)^2}{\kappa\,\rho\,(1+\kappa\,\rho)}+\frac{8\,m}{\hbar^2}\,\frac{U(\rho)}{\rho}, \tag{71}$$

where A is the integration constant.

By comparing Eq. (69) to Eq. (71), we obtain

$$\left(\frac{d\,\rho}{d\,\xi}\right)^2=A\,\rho-\frac{8\,m\,\epsilon}{\hbar^2}\rho^2+\frac{1}{\kappa}\left(\frac{2\,m\,u}{\hbar}\right)^2\frac{\rho}{1+\kappa\,\rho}+\frac{8\,m}{\hbar^2}\,\rho\,U(\rho). \tag{72}$$

The determination of the soliton shape is thus reduced to the solution of the first-order ordinary differential equation (72). By introducing the dimensionless variables

$$n=|\kappa|\,\rho \tag{73}$$

and

$$\chi=\frac{2\,m\,u}{\hbar}(x\mp u\,t), \tag{74}$$

the Eq. (72) takes the form

$$\left(\frac{d\,n}{d\,\chi}\right)^2=\frac{n}{\sigma+n}+\alpha\,n-\frac{2\,\epsilon}{m\,u^2}n^2+\frac{2\,|\kappa|}{m\,u^2}n\,\tilde{U}(n), \tag{75}$$

where $\alpha=A\,|\kappa|\,(\hbar/2\,m\,u)^2$ is an integration constant and $\tilde{U}(n)\equiv U(\rho)$. The parameter $\sigma=\kappa/|\kappa|$ assumes the value $+1$ when the inclusion principle holds $(\kappa>0, n\geqslant 0)$. Accordingly, for the exclusion principle ($\kappa<0, 0\leqslant n\leqslant 1$), we have $\sigma=-1$. We note, finally, that while solving Eq. (75), we have to take into account the two arbitrary constants α and ϵ that define a family of solutions.

From Eq. (57), we have, in the soliton case,

$$\langle \hat{p} \rangle = \pm M\, u, \tag{76}$$

where M is defined by

$$M = m \int_{-\infty}^{+\infty} \frac{\rho}{1+\kappa\,\rho}\, dx. \tag{77}$$

By using Eqs. (56), (26), (27), and (29), we can write the soliton energy

$$E = \frac{\langle \hat{p}^2 \rangle}{2\, m} + \frac{\kappa}{2} m\, u^2 \int_{-\infty}^{+\infty} \left[\left(\frac{\rho}{1+\kappa\,\rho} \right)^2 + U(\rho) \right] dx. \tag{78}$$

To evaluate $\langle \hat{p}^2 \rangle / 2\, m$, we take into account Eq. (61), and obtain

$$\frac{\langle \hat{p}^2 \rangle}{2\, m} = m\, u^2 \int_{-\infty}^{+\infty} \frac{\rho}{(1+\kappa\,\rho)^2}\, dx - \int_{-\infty}^{+\infty} \rho\, \frac{d\, U(\rho)}{d\rho}\, dx + \epsilon\, N. \tag{79}$$

Considering that $u = \langle \hat{p} \rangle / 2\, M$ and Eq. (79), the energy (78) satisfies the following soliton energy-momentum dispersion relation:

$$E = \frac{\langle \hat{p} \rangle^2}{2\, M} \left[1 + \int_{-\infty}^{+\infty} \frac{\rho}{(1+\kappa\,\rho)^2}\, dx \Bigg/ \int_{-\infty}^{+\infty} \frac{\rho}{1+\kappa\,\rho}\, dx \right] + \int_{-\infty}^{+\infty} \left[U(\rho) - \rho\, \frac{d\, U(\rho)}{d\rho} \right] dx + \epsilon\, N. \tag{80}$$

If we chose the constant ϵ, appearing in the phase of ψ, as

$$\epsilon\, N = \int_{-\infty}^{+\infty} \left[\rho\, \frac{d\, U(\rho)}{d\rho} - U(\rho) \right] dx - \frac{1}{2} m\, u^2 \int_{-\infty}^{+\infty} \frac{\rho}{(1+\kappa\,\rho)^2}\, dx, \tag{81}$$

the energy-momentum dispersion relation assumes the expression

$$E = \frac{\langle \hat{p} \rangle^2}{2\, M}, \tag{82}$$

which is related to a free particle of mass M, traveling with momentum $\langle \hat{p} \rangle = \pm M\, u$.

VII. EFFECTIVE POTENTIAL

In this section we prove that a unitary nonlinear transformation of the solitonic states $\psi(\xi,t)$ of Eq. (24) [$\psi(\xi,t) \rightarrow \phi(\xi,t)$] exists, and that the new states $\phi(\xi,t)$ are solutions of a Schrödinger equation with an algebraic real nonlinearity. In other words, we introduce a unitary transformation for the solitonic state $\psi(\xi,t)$ that reduces the derivative complex nonlinearity $\Lambda(\rho, \mathbf{j})$, introduced by the EIP, to an algebraic real one.

Let us consider the unitary transformations

$$\psi(\xi,\, t) \rightarrow \phi(\xi,\, t) = \mathcal{U}(\xi)\, \psi(\xi,\, t), \tag{83}$$

$$\mathcal{U}^{+}(\xi) = \mathcal{U}^{-1}(\xi), \tag{84}$$

where $\mathcal{U}(\xi)$ is given by

$$\mathcal{U}(\xi) = \exp\left\{ \frac{i}{\hbar} [\pm m\, u\, \xi - s(\xi)] \right\}. \tag{85}$$

We note that the transformation $\mathcal{U}$ actually is not well defined by Eq. (85) since it is only defined modulo integer multiples of $2\,\pi$. The new wave function $\phi(\xi,t)$ has the same amplitude of the wave function $\psi(\xi,t)$ but a different phase:

$$\phi(\xi,\, t) = \rho(\xi)^{1/2} \exp\left\{ \frac{i}{\hbar} (\pm m\, u\, \xi - \epsilon\, t) \right\}. \tag{86}$$

In Appendix B we show that the transformation defined by Eqs. (83) and (85), implies that the field $\phi(\xi,t)$ satisfies the following equation of motion:

$$i\,\hbar\, \frac{\partial\, \phi}{\partial\, t} = -\frac{\hbar^2}{2\, m} \frac{\partial^2\, \phi}{\partial\, x^2} + F_{\mathrm{eff}}(\rho)\, \phi, \tag{87}$$

where F_{eff} is given by

$$F_{\mathrm{eff}}(\rho) = F(\rho) + \frac{1}{2} m\, u^2\, \kappa\, \rho\, \frac{2+\kappa\,\rho}{(1+\kappa\,\rho)^2}. \tag{88}$$

By means of $F_{\mathrm{eff}}(\rho) = d\, U_{\mathrm{eff}}(\rho)/d\,\rho$, we can introduce the potential $U_{\mathrm{eff}}(\rho)$, which is given by

$$U_{\mathrm{eff}}(\rho) = U(\rho) + \frac{1}{2} m\, u^2\, \kappa\, \frac{\rho^2}{1+\kappa\,\rho}. \tag{89}$$

We remark that the term $(\kappa\, m\, u^2/2)\, \rho^2/(1+\kappa\,\rho)$ originates from $U_{\mathrm{EIP}}(\rho, \mathbf{j})$, and represent the EIP effect on the shape of the soliton.

Let us consider the transformation recently introduced by Doebner and Goldin [16,34]:

$$\psi(t,\, \mathbf{x}) \rightarrow \phi(t,\, \mathbf{x}) = \sqrt{\rho(t,\, \mathbf{x})} \exp\left[i \left(\frac{\gamma(t)}{2} \log \rho(t,\, \mathbf{x}) + \frac{\lambda(t)}{\hbar} S(t,\, \mathbf{x}) + \theta(t,\, \mathbf{x}) \right) \right]. \tag{90}$$

Transformation (90) defines a class of nonlinear gauge transformations, varying the parameters $\gamma(t)$, $\lambda(t)$, and $\theta(t,\mathbf{x})$, and has the important property of making linear a particular subfamily of equations belonging to the Doebner-Goldin equation family. By comparing Eq. (90) to Eq. (85), we can note that the transformation introduced in this work is a particular case of the more general transformation introduced by Doebner and Goldin. Transformation (85) is limited to the soliton states without linearizing the Schrödinger equation describing these states. It reduces the complex nonlinearity of the Schrödinger equation to another real one.

By following the same procedure of Sec. VI, we can determine the shape of the solitonic solutions of Eq. (87), that are the same states of Eq. (24) modulo the phase $S(\xi,t)$. We may obtain an equation analog to Eq. (75):

$$\left(\frac{d\,n}{d\,\chi}\right)^2=\alpha'\,n-\frac{2\,\epsilon'}{m\,u^2}n^2+\frac{2\,|\kappa|}{m\,u^2}n\,\tilde{U}_{\rm eff}(n), \tag{91}$$

where $\tilde{U}_{\rm eff}(n)=U_{\rm eff}(\rho)$. The parameters α' and ϵ' are related to that of Eq. (75) by $\epsilon=\epsilon'-m\,u^2/2$, $\alpha=\alpha'-\sigma$. The problem of searching the solitonic solutions of Eq. (24), with derivative complex nonlinearities due to the EIP, is now reduced to a search of the solitonic solutions of a Schrödinger equation with analytic real nonlinearity.

We consider now the nonlinear potential

$$U(\rho)=U_0(\rho)-\frac{1}{2}m\,u^2\,\kappa\frac{\rho^2}{1+\kappa\,\rho}, \tag{92}$$

where $U_0(\rho)$ is again an analytic real arbitrary potential in ρ, and the second term is selected with the scope of eliminating the effect of the EIP. Equation (75) now takes the form

$$\left(\frac{d\,n}{d\,\chi}\right)^2=\alpha'\,n-\frac{2\,\epsilon'}{m\,u^2}n^2+\frac{2\,|\kappa|}{m\,u^2}n\,\tilde{U}_0(n). \tag{93}$$

Equation (93) is identical to the equation of the solitonic shape that we may find in the literature if we take a NLSE with the analytic nonlinear potential $U_0(\rho)$ [33]. Then Eq. (93) allows us to use the soliton solutions of NLSE's available in literature.

VIII. APPLICATIONS

As a first application of the results obtained in Sec. VII, we consider the case where the potential $U(\rho)=0$. The particles of the system are submitted to the interaction introduced only by the EIP. Equation (91) has the same form as Eq. (75), with $U(\rho)=0$, and can be written as

$$\left(\frac{d\,n}{d\,\chi}\right)^2=\frac{n}{\sigma+n}+\alpha\,n+\beta\,n^2, \tag{94}$$

with $\beta=-2\,\epsilon/m\,u^2$. This is a first-order differential equation of the type $d\,n/d\,\chi=f(n)$ with $f(n)$ an analytic function, and can be integrated numerically. Alternatively, we can search analytic solutions of Eq. (94) by suitably choosing the values of the parameters α, β, and σ. Analytic one-soliton solutions of this equation were found in Ref. [35] in the particular cases $\sigma=1$, $\alpha=-1$, and $\beta=1$.

As a second application we now derive the nonlinear potential $U(\rho)$ which, when present simultaneously with the EIP potential $U_{\rm EIP}(\rho,\mathbf{j})$, permits the formulation of a soliton with shape given by $\rho(\xi)\propto[\cosh(b\,\xi)]^{-2}$. We start by considering the nonlinearity [36]

$$U_0(\rho)=-\frac{\mu}{2}\rho^2. \tag{95}$$

The Schrödinger equation with this nonlinearity was recently used to study the Bose-Einstein condensation [2–6]. In the Gross-Pitayevski equation the parameter μ is given by

$$\mu=\frac{4\,\pi\,\hbar^2\,N\,a}{m}, \tag{96}$$

where N is the number of atoms in the condensate, m their mass, and a is the s-wave triplet scattering length. Its value is assumed to range in the interval $85\,a_0<a<140\,a_0$, a_0 being the Bohr radius [37].

Setting $\alpha'=0$ and $\mu>0$, Eq. (93), with potential (95), is easily integrable, obtaining

$$\rho(\xi)=\frac{N\,b}{2}[\cosh(b\,\xi)]^{-2}, \tag{97}$$

where b is a dimensionless constant defined as

$$b=\frac{\mu\,m\,N}{2\,\hbar^2}, \tag{98}$$

and the condition of normalization $N=\int|\psi|^2\,d\xi$, that fixes the parameter $\epsilon=-\mu^2\,m\,N^2/8\,\hbar^2$, has been taken into account.

The phase $S(\xi,t)$ of the soliton takes the form

$$S(\xi,t)=-\epsilon\,t\pm m\,u\,\xi\mp m\,u\frac{c}{b}\tanh^{-1}[c\tanh(b\,\xi)], \tag{99}$$

with

$$c=\left(1+\frac{2}{\kappa\,b\,N}\right)^{-1/2} \tag{100}$$

a dimensionless constant.

The EIP effect modifies the phase of the soliton. In the limit $\kappa\to 0$, i.e., when the EIP is switched off, the phase of the soliton becomes equal to the phase of the soliton of the cubic Schrödinger equation.

Finally, we remark that in the case of a pure exclusion principle ($\kappa<0$), the soliton exists, as we can see from Eq. (99), only if

$$4\,\hbar^2>|\kappa|\,\mu\,m\,N^2. \tag{101}$$

If we take into account the maximum value of the quantity $\rho(\xi)$, that is $\rho(0)=\mu\,m\,N^2/4\,\hbar^2$ and the maximum number of particles that can be put in a site [21],

$$\rho_{\rm max}=\frac{1}{|\kappa|}, \tag{102}$$

Eq. (101) can be written in the form

$$\rho(0)<\rho_{\rm max}. \tag{103}$$

This imposes no violation of the exclusion principle in the central site, where the maximal occupation exists, and, consequently, no violation of the exclusion principle on all the other points of the space. Taking into account Eqs. (92) and

(95), we can write the potential $U(\rho)$, which generates the soliton given by Eqs. (97) and (99), as

$$U(\rho)=-\frac{\mu}{2}\rho^2-\kappa\frac{1}{2}m\,u^2\frac{\rho^2}{1+\kappa\,\rho}. \tag{104}$$

IX. CONCLUSIONS

Let us recall the main results obtained in this work. Equations (18) and (29) define the Lagrangian and the Hamiltonian density and, by varying the analytic potential $U(\rho)$, define a family of nonlinear quantum canonical systems obeying the EIP. We have also derived the Ehrenfest relations of these systems.

We have deduced Eq. (75) which is an ordinary differential equation of first order. It can be solved to determine the soliton shapes arised inside the system.

To make the system described by the Lagrangian (18) more realistic, in view of possible applications in condensed physics like superfluidity and superconductivity, it could be interesting to couple it with an Abelian or non-Abelian gauge field, like in the Chern-Simons model. This development is the argument of our future research.

APPENDIX A

In this appendix we show the validity of Eq. (55) for those systems described by the nonlinear real potential $U(\rho)+U_{\rm EIP}(\rho,\boldsymbol{\nabla}S)$ where $U_{\rm EIP}(\rho,\boldsymbol{\nabla}S)$ is the EIP potential written as a function of the density $\rho=\psi^*\psi$ and of the gradient of the phase $S=(i\hbar/2)\log(\psi^*/\psi)$. This potential can be obtained from Eqs. (20) and (37):

$$U_{\rm EIP}(\rho,\boldsymbol{\nabla}S)=\frac{\kappa}{2m}\rho^2(\boldsymbol{\nabla}S)^2. \tag{A1}$$

Equation (46), by posing $\hat{A}\equiv\hat{H}$, where $\hat{H}$ is given by Eq. (53), becomes

$$\frac{dE}{dt}=\int\left[-\frac{\hbar^2}{2m}\frac{\partial\psi^*}{\partial t}\Delta\psi-\frac{\hbar^2}{2m}\psi^*\Delta\frac{\partial\psi}{\partial t}+V\frac{\partial\rho}{\partial t}+\rho\frac{\partial V}{\partial t}+\frac{\partial}{\partial t}(U+U_{\rm EIP})\right]d^3x. \tag{A2}$$

By using the equations of motion of the fields ψ and ψ^*,

$$i\hbar\frac{\partial\psi}{\partial t}=\left[-\frac{\hbar^2}{2m}\Delta+V+F+W+i\,\mathcal{W}\right]\psi, \tag{A3}$$

$$-i\hbar\frac{\partial\psi^*}{\partial t}=\left[-\frac{\hbar^2}{2m}\Delta+V+F+W-i\,\mathcal{W}\right]\psi^*, \tag{A4}$$

with $\mathcal{W}$, F, and W given by Eqs. (3), (23), and (25), respectively, Eq. (A2) becomes

$$\begin{aligned}\frac{dE}{dt}=&\int\left[i\frac{\hbar^3}{4m^2}\Delta\psi^*\Delta\psi-i\frac{\hbar}{2m}\psi^*(V+F+W-i\,\mathcal{W})\Delta\psi\right]d^3x\\&-\int\left[i\frac{\hbar^3}{4m^2}\psi^*\Delta^2\psi+i\frac{\hbar}{2m}\psi^*\Delta[(V+F+W+i\,\mathcal{W})\psi]\right]d^3x\\&+\int\left[\left(V+\frac{\partial U}{\partial\rho}+\frac{\partial U_{\rm EIP}}{\partial\rho}\right)\frac{\partial\rho}{\partial t}+\frac{\partial U_{\rm EIP}}{\partial(\boldsymbol{\nabla}S)}\frac{\partial(\boldsymbol{\nabla}S)}{\partial t}\right]d^3x\\&+\left\langle\frac{\partial V}{\partial t}\right\rangle,\end{aligned} \tag{A5}$$

where we have used the relation

$$\frac{\partial}{\partial t}(U+U_{\rm EIP})=\frac{\partial U}{\partial\rho}\frac{\partial\rho}{\partial t}+\frac{\partial U_{\rm EIP}}{\partial\rho}\frac{\partial\rho}{\partial t}+\frac{\partial U_{\rm EIP}}{\partial(\boldsymbol{\nabla}S)}\frac{\partial(\boldsymbol{\nabla}S)}{\partial t}. \tag{A6}$$

In Eq. (A5), integrating by parts and neglecting the surface terms, we obtain

$$\begin{aligned}\frac{dE}{dt}=&\frac{-i\hbar}{2m}\int(V+F+W)(\psi^*\Delta\psi-\psi\Delta\psi^*)d^3x\\&-\frac{\hbar}{2m}\int\mathcal{W}(\psi^*\Delta\psi+\psi\Delta\psi^*)d^3x\\&+\int\left\{\left(V+\frac{\partial U}{\partial\rho}+\frac{\partial U_{\rm EIP}}{\partial\rho}\right)\frac{\partial\rho}{\partial t}-\boldsymbol{\nabla}\left[\frac{\partial U_{\rm EIP}}{\partial(\boldsymbol{\nabla}S)}\right]\frac{\partial S}{\partial t}\right\}d^3x+\left\langle\frac{\partial V}{\partial t}\right\rangle.\end{aligned} \tag{A7}$$

Using Eqs. (35) and (36), that we can rewrite in the forms

$$\frac{\partial\rho}{\partial t}=-\boldsymbol{\nabla}\left(\frac{\boldsymbol{\nabla}S}{m}\rho\right)+\frac{2}{\hbar}\rho\,\mathcal{W}, \tag{A8}$$

$$\frac{\partial S}{\partial t}=\frac{\hbar^2}{2m}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}-\frac{(\boldsymbol{\nabla}S)^2}{2m}-V-F-W, \tag{A9}$$

and taking into account the relations

$$\frac{\partial U}{\partial\rho}+\frac{\partial U_{\rm EIP}}{\partial\rho}=F+W, \tag{A10}$$

$$\boldsymbol{\nabla}\left[\frac{\partial U_{\rm EIP}}{\partial(\boldsymbol{\nabla}S)}\right]=-\frac{2}{\hbar}\rho\,\mathcal{W}, \tag{A11}$$

$$-\frac{i\hbar}{2m}(\psi^*\Delta\psi-\psi\Delta\psi^*)=\boldsymbol{\nabla}\left(\frac{\boldsymbol{\nabla}S}{m}\rho\right), \tag{A12}$$

$$(\psi^* \Delta \psi + \psi \Delta \psi^*) = 2\rho\left[\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} - \left(\frac{\boldsymbol{\nabla} S}{\hbar}\right)^2\right] \quad \text{(A13)}$$

derived from Eqs. (A1) and (34), Eq. (A7) becomes

$$\begin{aligned}\frac{dE}{dt} &= \int (V+F+W)\,\boldsymbol{\nabla}\left(\frac{\boldsymbol{\nabla} S}{m}\rho\right) d^3x \\ &\quad - \frac{\hbar}{m}\int \rho\,\mathcal{W}\left[\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} - \left(\frac{\boldsymbol{\nabla} S}{\hbar}\right)^2\right] d^3x \\ &\quad + \int (V+F+W)\left[-\boldsymbol{\nabla}\left(\frac{\boldsymbol{\nabla} S}{m}\rho\right) + \frac{2}{\hbar}\rho\,\mathcal{W}\right] d^3x \\ &\quad + \frac{2}{\hbar}\int \rho\,\mathcal{W}\left[\frac{\hbar^2}{2m}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} - \frac{(\boldsymbol{\nabla} S)^2}{2m} - V - F - W\right] d^3x \\ &\quad + \left\langle \frac{\partial V}{\partial t}\right\rangle. \end{aligned} \quad \text{(A14)}$$

We can immediately obtain

$$\frac{dE}{dt} = \left\langle \frac{\partial V}{\partial t}\right\rangle, \quad \text{(A15)}$$

which coincides with Eq. (55).

APPENDIX B

In this appendix we show that the unitary transformation introduced by Eqs. (83) and (85), in the case of soliton states, transform Eq. (24), containing a complex nonlinearity, into Eq. (87), containing a real nonlinearity. Let us consider definitions (83) and (85) and Eq. (66). The unitary transformation can be rewritten as

$$\psi(\xi,t) = \exp\left[-\frac{i}{\hbar}\Gamma(\xi)\right]\phi(\xi,t), \quad \text{(B1)}$$

where

$$\Gamma(\xi) = \pm m u\left(\xi - \int_0^{\xi}\frac{d\xi'}{1+\kappa\rho(\xi')}\right), \quad \text{(B2)}$$

and $\xi = x \mp u t$. After deriving Eq. (B1), we obtain the relations

$$\frac{\partial\psi}{\partial t} = \left[\frac{\partial\phi}{\partial t} - \frac{i}{\hbar}\frac{\partial\Gamma}{\partial t}\phi\right] e^{-i\Gamma/\hbar}, \quad \text{(B3)}$$

$$\begin{aligned}\frac{\partial^2\psi}{\partial\xi^2} &= \left[-\frac{i}{\hbar}\frac{\partial^2\Gamma}{\partial\xi^2}\phi - \frac{1}{\hbar^2}\left(\frac{\partial\Gamma}{\partial\xi}\right)^2\phi \right.\\ &\quad \left. - i\frac{2}{\hbar}\frac{\partial\Gamma}{\partial\xi}\frac{\partial\phi}{\partial\xi} + \frac{\partial^2\phi}{\partial\xi^2}\right] e^{-i\Gamma/\hbar}. \end{aligned} \quad \text{(B4)}$$

By considering Eqs. (60), (65), and (B2), we obtain

$$\frac{\partial S}{\partial\xi} = \frac{1}{\kappa\rho}\frac{\partial\Gamma}{\partial\xi}, \quad \text{(B5)}$$

and then the current $\mathbf{j}$, given by Eq. (37), becomes

$$\mathbf{j} = \frac{1}{\kappa m}(1+\kappa\rho)\frac{\partial\Gamma}{\partial\xi}. \quad \text{(B6)}$$

In the case of soliton states the Schrödinger equation (24) becomes

$$\begin{aligned} i\hbar\frac{\partial\psi}{\partial t} &= -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial\xi^2} + F(\rho)\psi + \kappa\frac{m}{\rho}\left(\frac{\mathbf{j}}{1+\kappa\rho}\right)^2\psi \\ &\quad - i\kappa\frac{\hbar}{2\rho}\frac{\partial}{\partial\xi}\left(\frac{\mathbf{j}\rho}{1+\kappa\rho}\right)\psi. \end{aligned} \quad \text{(B7)}$$

By using Eqs. (B3), (B4), and (B6), we may write Eq. (B7) in the form

$$\begin{aligned}\frac{\partial\Gamma}{\partial t}\phi + i\hbar\frac{\partial\phi}{\partial t} &= -\frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial\xi^2} + \frac{i\hbar}{2m}\frac{\partial^2\Gamma}{\partial\xi^2}\phi + F(\rho)\phi \\ &\quad + \frac{2+\kappa\rho}{2m\kappa\rho}\left(\frac{\partial\Gamma}{\partial\xi}\right)^2\phi + \frac{i\hbar}{m}\frac{\partial\Gamma}{\partial\xi}\frac{\partial\phi}{\partial\xi} \\ &\quad - \frac{i\hbar}{2m\rho}\frac{\partial}{\partial\xi}\left(\frac{\partial\Gamma}{\partial\xi}\rho\right)\phi. \end{aligned} \quad \text{(B8)}$$

We use now the relation $\partial\Gamma/\partial t = \mp u\,\partial\Gamma/\partial\xi$ with

$$\frac{\partial\Gamma}{\partial\xi} = \pm m u\frac{\kappa\rho}{1+\kappa\rho}, \quad \text{(B9)}$$

[easily derivable from Eqs. (65) and (B6)], Eq. (B8), and

$$\begin{aligned} -\frac{i\hbar}{2m\rho}\frac{\partial\Gamma}{\partial\xi}\frac{\partial\rho}{\partial\xi}\phi + \frac{i\hbar}{m}\frac{\partial\Gamma}{\partial\xi}\frac{\partial\phi}{\partial\xi} &= \frac{i\hbar}{2m}\left(\phi^*\frac{\partial\phi}{\partial\xi} - \phi\frac{\partial\phi^*}{\partial\xi}\right)\frac{\partial\Gamma}{\partial\xi}\frac{\phi}{\rho}. \end{aligned} \quad \text{(B10)}$$

If we take into account that $\mathbf{j}_\phi = \pm u\rho$, we arrive at the following NLSE:

$$i\hbar\frac{\partial\phi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial\xi^2} + F(\rho)\phi + \frac{1}{2}m u^2\kappa\rho\frac{2+\kappa\rho}{(1+\kappa\rho)^2}\phi. \quad \text{(B11)}$$

Let us now introduce the variable x and define $F_{\mathrm{eff}}(\rho)$:

$$F_{\mathrm{eff}}(\rho) = F(\rho) + \frac{1}{2}m u^2\kappa\rho\frac{2+\kappa\rho}{(1+\kappa\rho)^2}; \quad \text{(B12)}$$

then Eq. (B11) can be rewritten as

$$i\hbar\frac{\partial\phi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial x^2} + F_{\mathrm{eff}}(\rho)\phi. \quad \text{(B13)}$$

Equation (B13) coincides with Eq. (87).

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